In my note, it is stated that "Eq. (10) suggests the possibility of defining a Mach number, which becomes one at choking for $0 \leqslant \alpha \leqslant 1$." The corresponding speed of sound can be obtained from Eq. (10):

$$C_e = [\Gamma(5k/3m)(1 + \alpha)T]^{1/2} \tag{1}$$

where

$$\Gamma = \frac{1 + [(\gamma - 1)/\gamma](2.5 + t_i)F}{1 + (\gamma - 1)(1.5 + t_i)F}$$

with

$$F = \frac{t_i \alpha \chi}{2 + (2 + 1.5\chi + \chi t_i)} \qquad \chi = \frac{2(1 - \alpha)}{2 - \alpha}$$

By taking $\gamma = \frac{5}{3}$ for monatomic gases and by performing some algebraic manipulations, the expression Γ can be written in the following form:

$$\Gamma = \frac{[5/\alpha(1 - \alpha)] + (2.5 + t_i)^2}{\frac{3}{2}[(1 + \alpha)(2 - \alpha)/\alpha(1 - \alpha)] + (1.5 + t_i)^2} \tag{2}$$

This expression agrees with γ^* given by Eq. (12) obtained by Jones and McChesney.[2] Therefore, C_e given in Eq. (1) is indeed the equilibrium speed of sound. This fact also establishes the correctness of the basic equations used in my note.

Although in my note the correct equilibrium speed of sound was obtained, it was not identified as such. To be consistent, the equilibrium speed of sound should be used for equilibrium flows. This appears to be the point in McChesney's comments.†

The fact that equilibrium speed of sound comes out automatically from the basic equations and the condition of choking suggests the following point: the frozen speed of sound may be used even for equilibrium flows provided some care is used in interpreting the results. Finally, it is to be noted that, for nonequilibrium flows, a meaningful speed of sound cannot be defined such that choking will occur at Mach number equal to 1.

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† Choking at equilibrium Mach number 1 has been pointed out earlier to the author by David C. Dryburgh of Rolls-Royce Limited, England.

Transverse Curvature Effects in Axisymmetric Hypersonic Boundary Layers

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It has been shown that mass transfer affects the induced surface pressure resulting from shock-wave boundary-layer interaction on a flat plate.[1, 2] A logical extension of this problem is to broaden these considerations to include axisymmetric configurations that would require the retention of transverse curvature terms in the boundary-layer equations. The authors have considered this problem, and the results will be presented in Ref. 3. In matching the inviscid and viscous flow solutions, it is necessary to investigate the effects of surface blowing on the vertical velocity component at the boundary between the inviscid and viscous flow regions. In this note, the expression for v_e/u_e will be derived. This result is compared with the recently published formula of Thyson and Schurmann.[4]

Consider the continuity equation for a nonreacting gas flowing on the surface of an axisymmetric body defined by $r_w = r_w(x)$:

$$(\partial/\partial x)(\rho u r) + (\partial/\partial y)(\rho v r) = 0 \tag{1}$$

where $r = r_w + y\cos\alpha$ (Fig. 1). Integration of Eq. (1) with respect to y yields

$$\int_0^\delta \frac{\partial}{\partial x}(\rho u r)dy + \rho_e v_e r_e - \rho_w v_w r_w = 0 \tag{2}$$

where $r_e = r_w + \delta\cos\alpha$. The integral term in the foregoing equation can be rewritten as

$$\int_0^\delta \frac{\partial}{\partial x}(\rho u r)dy = \frac{d}{dx}\int_0^\delta \rho u r dy - \rho_e u_e r_e \frac{d\delta}{dx} \tag{3}$$

Equation (2) thus becomes

$$\frac{d}{dx}\int_0^\delta \rho u r dy - \rho_e u_e r_e \frac{d\delta}{dx} + \rho_e v_e r_e - \rho_w v_w r_w = 0 \tag{4}$$

The definition of the displacement thickness δ^* for axisymmetric boundary-layer flow[5] is

$$\int_0^{\delta^*} \rho_e u_e r dy = \int_0^\delta (\rho_e u_e - \rho u) r dy \tag{5}$$

It follows that

$$\int_0^\delta \rho u r dy = \int_{\delta^*}^\delta \rho_e u_e r dy \tag{6}$$

Combining Eqs. (4) and (6) yields

$$\frac{d}{dx}\int_{\delta^*}^\delta \rho_e u_e r dy - \rho_e u_e r_e \frac{d\delta}{dx} + \rho_e v_e r_e - \rho_w v_w r_w = 0 \tag{7}$$

The integral term in Eq. (7) can easily be computed as follows:

$$\int_{\delta^*}^\delta \rho_e u_e (r_w + y)dy = \rho_e u_e r_w(\delta - \delta^*) + \rho_e u_e \frac{1}{2}(\delta^2 - \delta^{*2}) \tag{8}$$

where, for a very slender body of revolution, $\cos\alpha \cong 1$. Combining Eqs. (7) and (8) yields

$$\rho_e v_e r_e = \rho_w v_w r_w + \rho_e u_e r_e \frac{d\delta}{dx} - \frac{d}{dx}\left[\rho_e u_e r_w(\delta - \delta^*) + \rho_e u_e \frac{1}{2}(\delta^2 - \delta^{*2})\right] \tag{9}$$

which can also be written as follows:

$$\frac{v_e}{u_e} = \frac{\rho_w v_w r_w}{\rho_e u_e r_e} + \frac{d\delta}{dx}\left(1 - \frac{r_w}{r_e}\right) + \frac{r_w}{r_e}\left[\frac{d\delta^*}{dx} - \frac{\delta - \delta^*}{\rho_e u_e r_w}\frac{d}{dx}(\rho_e u_e r_w)\right] - \frac{1}{2\rho_e u_e r_e}(\delta^2 - \delta^{*2})\frac{d}{dx}(\rho_e u_e) - \frac{1}{r_e}\left(\delta\frac{d\delta}{dx} - \delta^*\frac{d\delta^*}{dx}\right) \tag{10}$$

The corresponding formula given in Ref. 4 is as follows:

$$\frac{v_e}{u_e} = \frac{\rho_w v_w r_w}{\rho_e u_e r_e} + \frac{d\delta}{dx}\left(1 - \frac{r_w}{r_e}\right) + \left[\frac{d\bar{\delta}^*}{dx} - \frac{\delta - \bar{\delta}^*}{\rho_e u_e r_w}\frac{d}{dx}(\rho_e u_e r_w)\right] \tag{11}$$

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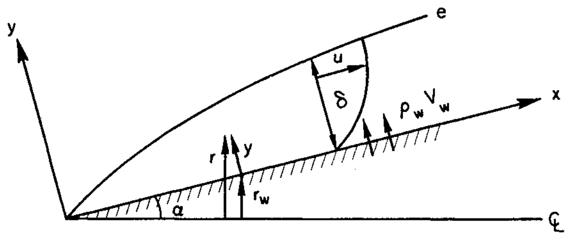

Fig. 1 Coordinate system.

It should be noted that Thyson and Schurmann[4] used a two-dimensional displacement thickness δ^*, and their formula does not fully account for the transverse curvature effect. In hypersonic boundary layers, $\delta^* \to \delta$; then, Eq. (10) becomes

$$\frac{v_e}{u_e} = \frac{\rho_w v_w r_w}{\rho_e u_e r_e} + \frac{d\delta^*}{dx} \tag{12}$$

Equation (12) provides the necessary matching condition used in Ref. 3.

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Reply by Authors to T.-Y. Li and J. F. Gross

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If transverse curvature effects are to be included, the inviscid streamline slope expression as given by Li and Gross is quite correct. In our original note, a transverse curvature effect was omitted which multiplies the rate of changes of the displacement thickness and inviscid flow. The slope is

$$\frac{V_e}{U_e} = \frac{\rho_w V_w}{\rho_e U_e}\left[\frac{1}{1+(\delta/r)}\right] + \frac{d\delta}{dx}\left\{1 - \left[\frac{1}{1+(\delta/r)}\right]\right\} + \left[\frac{d\delta^*}{dx} - \frac{\delta - \delta^*}{\rho_e U_e r}\frac{d}{dx}(\rho_e U_e r)\right]\left[\frac{1}{1+(\delta/r)}\right]$$

The difference then between the expression as given by Li and Gross and the foregoing expression is that they have properly included a transverse curvature effect in the axisymmetry displacement thickness. However, our displacement thickness is an axisymmetric displacement thickness. What we neglected was a transverse curvature effect on the displacement thickness. Since we did not consider transverse effects, our pressure interaction analysis as such is still valid.

Received June 2, 1964.
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Comment on "Orbital Motion in the Theory of General Relativity"

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In their discussion[1] of relativistic orbital mechanics, Anderson and Lorell note the difficulty in interpreting the equations of motion. They go on to mention that, although their result for the secular change in the argument of pericenter agrees with that of Bogorodskii,[2] they do not match the latter in regard to his estimate of the effect on χ (i.e., $-n\tau$). The difficulty lies in the fact that, in the two papers, somewhat different perturbative components, R and S (the former radial, the latter tangential), are used. Further, Bogorodskii's error in computing changes in χ was not corrected, as is evident in Eq. (4) of Ref. 1.

That this equation is incomplete has been noted before,[3] although a very careful reading of the source material[4] is required in order to perceive this; other sources[5,6] are only slightly less ambiguous.

We find, employing the same notation as Anderson and Lorell, and following Brouwer and Clemence[6] (especially pages 285–286 and 300–301),

$$\overline{d\chi/dt} = [3\mu n/ac^2(1-e^2)] \times [2 + e^2 - (3 + 2e + e^2 + 2e^3)/(1-e^2)]$$

Incidentally, utilizing Bogorodskii's perturbations (but setting his ω_0 equal to zero), we obtain

$$\overline{d\chi/dt} = [3\mu n/c^2a(1-e^2)^{1/2}] \times [2 - 2e^2 - (5 + 7e + 2e^2 + 3e^3 + 3e^4)/(1-e^2)^{3/2}]$$

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Received July 18, 1963.
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Reply by Author to H. R. Westerman

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The comment by H. R. Westerman is exemplary of a certain amount of confusion that has resulted from the introduction of the mean longitude L in the relativistic perturbations of Ref. 1. The purpose of this note is to clarify the meaning of the averaged rate in the mean longitude.

When the time rate of change of the mean anomaly phase angle χ is defined as in Eq. (4) of Ref. 1, then the mean anomaly rate is given by

$$dM/dt = n + (d\chi/dt) \tag{1}$$

Received June 16, 1964.
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